ILLUSTRATIVE SOLUTION OF A MIXED PROBLEM OF STEADY-STATE HEAT-CONDUCTION THEORY FOR A HALF-PLANE WITH A BOUNDARY CONDITION OF THE THIRD KIND

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A mixed boundary-value problem of potential theory is treated for a half-plane in the case in which two equal parts of the boundary are at constant but different temperatures, while there is heat transfer in accordance with Newton's law over the rest of the surface.

Formulation of the Problem

We are to find the solution of the Laplace equation

$$\Delta u = 0,\ y > 0,\ -\infty < x < +\infty \tag{1}$$

under the boundary conditions

$$u = T_1,\ y = 0,\ -1 \leqslant x \leqslant -\alpha,\ u = T_2,\ y = 0,\ \alpha \leqslant x \leqslant 1, \tag{2}$$

$$-\frac{\partial u}{\partial y} + hu = 0,\ y = 0,\ |x| < \alpha,\ |x| > 1 \tag{3}$$

and conditions at infinity,

$$|u| < \frac{A}{r},\ \left|\frac{\partial u}{\partial x}\right| < \frac{A}{r^2},\ \left|\frac{\partial u}{\partial y}\right| < \frac{A}{r^2},\ r = \sqrt{x^2 + y^2} \to \infty,$$

where A, $h > 0$, T_1, and T_2 are constants, and we have $0 < \alpha < 1$. We seek a solution of problem (1)-(3) as the sum of two functions,

$$u = u^+ + u^-, \tag{4}$$

where

$$u^+(x;\ y) = u^+(-x;\ y),\ u^-(-x;\ y) = -u^-(x;\ y).$$

Each of the terms in (4) satisfies Eq. (1) and condition (3); condition (2) is written

$$u^\pm = T^\pm,\ y = 0,\ \alpha \leqslant x \leqslant 1,$$

where

$$T^+ = \frac{T_2 + T_1}{2},\ T^- = \frac{T_2 - T_1}{2}. \tag{5}$$

To solve problem (1) we introduce the complex potential

$$\Phi(z) = f^+(z) + f^-(z),\ z = x + iy,\ y \geqslant 0, \tag{6}$$

where*

$$f^\pm(z) = u^\pm + iv^\pm.$$

*Below we omit the "±."

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Integral Fredholm Equations for the Imaginary Part of the Complex Potential

Using the result of [2] we can easily find integral equations for the functions $v^+(x)$ and $v^-(x)$ [3]:

$$v^+(x) + \frac{h}{\pi}\int_\alpha^1 v^+(t)\ln\left|\frac{(1-\alpha^2)(x^2-t^2)}{[\sqrt{(t^2-\alpha^2)(1-x^2)}+\sqrt{(x^2-\alpha^2)(1-t^2)}]^2}\right| dt =$$

$$= hT^+(1-\alpha)\left[\frac{1-x}{1-\alpha} - \frac{2}{\pi}\arcsin\sqrt{\frac{1-x^2}{1-\alpha^2}}\right] +$$

$$+ v^+(\alpha)\frac{2}{\pi}\arcsin\sqrt{\frac{1-x^2}{1-\alpha^2}} + v^+(1)\frac{2}{\pi}\arccos\sqrt{\frac{1-x^2}{1-\alpha^2}}, \tag{7}$$

where $\alpha \le x \le 1$, $v^+(-x) = -v^+(x)$.

$$v^-(x) + \frac{h}{\pi}\int_\alpha^1 v^-(t)\ln\left|\frac{H[F(\varphi;\ k) - F(\psi;\ k)]}{H[F(\varphi;\ k) + F(\psi;\ k)]}\right| dt =$$

$$= hT^-(1-\alpha)\left[\frac{1-x}{1-\alpha} - \frac{F(\psi;\ k)}{K}\right] + v^-(\alpha)\frac{F(\psi;\ k)}{K} + v^-(1)\left[1 - \frac{F(\psi;\ k)}{K}\right], \tag{8}$$

$$\alpha \leqslant x \leqslant 1,\ v^-(x) = v^-(-x),$$

where

$$\varphi = \arcsin\sqrt{\frac{1-t^2}{1-\alpha^2}},\quad \psi = \arcsin\sqrt{\frac{1-x^2}{1-\alpha^2}},$$

$F(\varphi;k)$ is the elliptic integral of the first kind of modulus $k = \sqrt{1-\alpha^2}$, $k' = \alpha$; $K = K(k)$ is the complete elliptic integral, and $H(W)$ is the Jacobi eta function.

We seek solutions of Eqs. (7) and (8) in the form

$$v(x) = v_0(x) + v(\alpha)A(x) + v(1)B(x). \tag{9}$$

The functions $v_0(x)$, $A(x)$, and $B(x)$ are found by the method of successive approximations as series in powers of h [1]. The complex potential (6) is determined within the constant quantities $v(\alpha)$ and $v(1)$:

$$f(z) = f_0(z) + v(\alpha)f_1(z) + v(1)f_2(z). \tag{10}$$

Here

$$f_0(z) = hT(1-\alpha)\gamma(z) + h\int_\alpha^1 v_0(t)G(z;\ t)\,dt;$$

$$f_1(z) = -\gamma(z) + h\int_\alpha^1 A(t)G(z;\ t)\,dt;$$

$$f_2(z) = \gamma(z) + h\int_\alpha^1 B(t)G(z;\ t)\,dt;$$

$$\gamma^+(z) = \frac{2}{\pi}\int_z^{i\infty}\frac{\xi\exp hi(\xi-z)\,d\xi}{\sqrt{(\xi^2-1)(\xi^2-\alpha^2)}};$$

$$\gamma^-(z) = \frac{1}{K}\int_z^{i\infty}\frac{\exp hi(\xi-z)\,d\xi}{\sqrt{(\xi^2-1)(\xi^2-\alpha^2)}};$$

$$G^+(z;\ t) = \frac{2}{\pi}\sqrt{(1-t^2)(t^2-\alpha^2)}\int_z^{i\infty}\frac{\xi\exp hi(\xi-z)\,d\xi}{(t^2-\xi^2)\sqrt{(\xi^2-1)(\xi^2-\alpha^2)}};$$

$$G^-(z;\ t) = \frac{2t}{\pi}\sqrt{(1-t^2)(t^2-\alpha^2)}\int_z^{i\infty}\frac{\exp hi(\xi-z)\,d\xi}{(t^2-\xi^2)\sqrt{(\xi^2-1)(\xi^2-\alpha^2)}} - \frac{2K}{\pi}Z(\varphi;\ k)\gamma^-(z),\quad \operatorname{Im} z \geqslant 0;$$

$Z_*(\varphi; k) = E(\varphi; k) - (E/K)F(\varphi; k)$ is the zeta function, $E(\varphi; k)$ is the elliptic integral of the second kind, and $E = E(k)$ is the complete elliptic integral. Setting $z = \alpha$ in (10), we find a system of algebraic equations for the constants $v(\alpha)$ and $v(1)$:

$$\begin{aligned} \operatorname{Re} f(\alpha) &= \operatorname{Re} f_0(\alpha) + v(\alpha) \operatorname{Re} f_1(\alpha) + v(1) \operatorname{Re} f_2(\alpha), \\ \operatorname{Im} f(\alpha) &= \operatorname{Im} f_0(\alpha) + v(\alpha) \operatorname{Im} f_1(\alpha) + v(1) \operatorname{Im} f_2(\alpha). \end{aligned} \tag{11}$$

Solving system (11) by the method of successive approximations, we find expansions for $v(\alpha)$ and $v(1)$ which conv(rge at sufficiently small values of h:

$$\frac{v^+(\alpha)}{T^+} = -h\alpha + h \frac{K' - E'}{\left[\ln \frac{2}{h\sqrt{1-\alpha^2}} - C\right]} + O(h^2),$$

$$\frac{v^+(1)}{T^+} = -h + \frac{\frac{\pi}{2} + h(K' - E')}{\left[\ln \frac{2}{h\sqrt{1-\alpha^2}} - C\right]} - \frac{2hE_1(k)}{\left[\ln \frac{2}{h\sqrt{1-\alpha^2}} - C\right]^2} + O(h^2),$$

$$\frac{v^-(\alpha)}{T^-} = -\frac{K}{K'} + \frac{h}{K'}\left[\ln \frac{2e}{h\sqrt{1-\alpha^2}} - C\right] - h\left[\alpha + \frac{2F_1(k)}{(K')^2}\right] + O(h^2 \ln h), \tag{12}$$

$$\frac{v^-(1)}{T^-} = \frac{h}{K'}\left[\ln \frac{2e}{h\sqrt{1-\alpha^2}} - C\right] - h + O(h^2 \ln h),$$

where

$$E_1(k) = \int_0^{\pi/2} E(\varphi;\ k)\, d\varphi; \quad F_1(k) = \int_0^{\pi/2} F(\varphi;\ k)\, d\varphi; \quad K' = K(k'),$$

and $C = 0.577215\ldots$ is the Euler constant.

Using expansions (12), we can directly calculate the total heat fluxes across the various parts of the boundary: $[-\alpha, +\alpha]$, $[-1; +1]$, $[\alpha; 1]$, $[1; \infty)$. For example, the total heat flux across part $[\alpha; 1]$ is

$$\begin{aligned} \frac{Q}{K_0} = \frac{T_2 + T_1}{2} &\left\{ \frac{\frac{\pi}{2}}{\left[\ln \frac{2}{h\sqrt{1-\alpha^2}} - C\right]} - \right. \\ &\left. - \frac{2hE_1(k)}{\left[\ln \frac{2}{h\sqrt{1-\alpha^2}} - C\right]^2} + O(h^2) \right\} + \frac{T_2 - T_1}{2}\left\{\frac{K}{K'} + \right. \\ &\left. + \frac{2hF_1(k)}{(K')^2} + O(h^2 \ln h)\right\} - hT_2(1 - \alpha), \end{aligned} \tag{13}$$

where K_0 is the thermal conductivity and Q is the total heat flux.

We note, in conclusion, that the nature of the analytic dependence of the solutions on the parameter h in this problem is quite different from that in the classical case of [1].

LITERATURE CITED

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